Monopoles and Gravity

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Submitted in partial fulfilment of the requirement of the degree of Master of Science in

Quantum Fields and Fundamental Forces

Imperial College London

September 2023

Acknowledgements

I would like to thank Prof João Magueijo for suggesting this topic and agreeing to supervise me, my QFFF friends who accompanied me throughout this special year, and Dr Tom Złośnik who introduced me to the concepts in chapter 5.

Abstract

We review the principal ideas behind the literature on the topic of gravity monopoles, which are defined as analogous to the well-studied electromagnetic ones. After introducing the latter, we cover the former in the linearized regime, as well as a general way to obtain their corresponding quantization condition, as a spin-2 gauge theory. The duality-invariant interpretation of the Taub-NUT solution to Einstein gravity is then presented, and its relation with the previous quantization condition. Lastly, concepts from Cartan geometry are introduced and used to show a possible condition for which the Bianchi identity is not satisfied, in a spontaneously broken $SO(1,4)$ gauge theory, in a manner akin to the 't Hooft-Polyakov monopole.

Contents

Chapter 1

Introduction

1.1 Electromagnetism

In 1931 P.A.M. Dirac published[1] a paper postulating his famous equation for the dynamics of a spin-1/2 particle, which predicted the existence of a new kind of matter called antimatter. This revolutionary formula stemmed from his philosophy of discovering new physics by seeking beauty and symmetry in nature's equations. A little less known, is the fact that it was only a few years later when he published another scientific article, the consequence of which he was even more excited about. This paper was on the nature of what is called the magnetic monopole.

Since Maxwell's famous formulation of the equations of electrodynamics, which describe the behaviour of electric and magnetic fields, it was noted that there was an unaesthetic asymmetry about them. Although sources of single electric charge, or monopoles, had already been detected in nature (e.g. electrons for negative and protons for positive charge), not a single source of either positive or negative "magnetic charge" had - no matter how close we look at any source of magnetic field, we always find north and south poles[1] in pairs, such that overall the charge density is zero. That is, we always detect dipoles. The consequence of this being that even though we have a term in Maxwell's equations for the electric charge density and current, ρ_e and $\mathbf{j}_e$, we don't have terms for the magnetic charge density and

[1] North and south poles are historical names analogous to positive and negative magnetic charge.

current, ρ_m and $\mathbf{j}_m$. Therefore, under the exchange of electric and magnetic fields the equations are clearly not symmetric. As well, even though these new magnetic terms could be added artificially, the fact that magnetic fields are described by a vector potential gave rise to purely geometric reasons which seemed to further impose that this density (and thus current) had to be zero.

In his new paper, Dirac considered an infinite number of vertically aligned magnets. The first would have its north pole at the origin, and its south pole right below. The next would also be vertical, with its north pole right below the south pole of the previous. If we continue this ad infinitum, and make each magnet infinitely small, while still keeping each of them infinitely close to the next, the opposite poles cancel out everywhere except at the origin. At this point, it looks like there is an isolated north pole at the origin, a magnetic monopole, while the line of infinitely small magnets becomes a singularity that cannot be observed, also called the Dirac string singularity.

Thanks to this incredible new mathematical trick, Dirac found a solution for the vector potential of a magnetic field that gives a non-zero magnetic charge density at the origin! With such an exact solution, there was motivation to include the extra magnetic density and current, which in turn made Maxwell's equations perfectly symmetric under exchange of the electric and magnetic fields. But there is more, because when he quantized the system another direct consequence appeared: the electric charge must be quantized! This is known as the Dirac quantization condition, and it states that if a single magnetic monopole exists in the universe, then all electric charge must be quantized. Given that electric charge seems to come in blocks in the universe, this would be a fascinating reason as to why.

Although the symmetry introduced to electrodynamics due to the inclusion of magnetic monopoles was enough for Dirac, it is also a valid point of view that just because a magnetic monopole can be added to the equations, it doesn't mean it should. However, in 1974 't Hooft[2] showed that a gauge theory with a compact covering Lie group, coupled to a scalar field, leads to the existence of non-singular topological solitons which look like a magnetic monopole from far away. Due to our understanding of particle physics and the standard model, we know that electromagnetism is obtained from a higher gauge theory, $SU(2) \times U(1)_Y$, which is broken to $U(1)_e$. Although the current standard model group $SU(3) \times SU(2) \times U(1)_Y$ is not compact, it is expected that the gauge group of a more complete "grand unified theory", such as $SU(5)$ or $SO(10)$, will be. Since this topological soliton, called a 't Hooft-Polyakov monopole, is also known to be equivalent to the Dirac monopole, physicists around the world have been expecting these monopoles to be detected in nature ever since. However, even to this day, no-one has been able to find them. How has such a fundamental piece of one of our most well-understood theories been able to elude us for so long?

This topic is introduced in chapter 2.

1.2 Gravity

For a lot of physicists, one of the earliest memories from our physics education, is pairing Newton's equation of gravity and Coulombs Law for electrostatics side by side:

$$F_N = -G\frac{m_1 m_2}{r^2} \quad || \quad F_C = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \tag{1.1}$$

and noticing the eerie similarity between them (granted, up to a constant). More to the fact, if we consider the next-order corrections to Newtonian gravity, due to Einstein's theory of general relativity, we again obtain a theory which looks highly similar to those from Maxwell's electrodynamics!

Among the plethora of questions regarding the nature of both theories that this has raised over time, and which are yet to be solved, a natural one to ask seems to be whether Dirac's monopole solution could be found in gravity, given its renewed similarity with electrodynamics, and what the consequences would be. This is at the heart of what this dissertation will review.

A. Zee considered[3] the existence of such a solution in gravity, for this linearized (weak field) limit. He speculated on its consequences, like all mass (later we find its energy) being quantized in a manner analogous to the Dirac quantization condition, as well as what the place of such a solution in the complete theory could be. This is covered in the beginning of chapter 3, along with a full general treatment of the quantization condition for a spin-2 gauge theory (which is equivalent to linearized gravity) that includes magnetic monopoles.

It is important to reiterate, that the similarity between gravity and electromagnetism is only explicit at the linear level. However, via a process of many papers and several authors, a solution to the non-linear Einstein field equations was found, called Taub-NUT. It can be interpreted as corresponding to the spacetime surrounding a dyon - an object with mass and its dual, "magnetic" mass. In the linearized limit, Taub-NUT reduces to Zee's gravipole. This is covered in chapter 4 along with its implication on the mass quantization condition.

Finally, in chapter 5 we give a brief overview of how Cartan geometry can be used to describe gravity as a spontaneously broken $SO(1, 4)$ gauge theory. It is shown how this can be used to replicate the 't Hooft-Polyakov monopole of an $SO(3)$ gauge theory, and extended to give the conditions in which the broken gravity theory could yield analogous gravipoles.

Very interesting efforts are going into finding experimentally verifiable signatures of rotating black holes with dual mass[4, 5], for which the solution is known as Kerr-Newman-Taub-NUT. As well as analysing their thermodynamic properties[6].

In the future, uncovering the problem of the magnetic monopole in gravity could teach us about the nature of this force, energy, electromagnetism and how they all fit with the rest of the universe.

Chapter 2

The electromagnetic monopole

2.1 Dirac monopole

Maxwell's equations of electromagnetism can be written in covariant form as $\partial_\mu F^{\mu\nu} = j^\nu$ with the Bianchi identity $\partial_\mu \tilde{F}^{\mu\nu} = 0$, where the dual field strength is $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$. We can make these equations more symmetric by introducing a new current in the Bianchi identity, such that $\partial_\mu \tilde{F}^{\mu\nu} = \tilde{j}^\nu$ and this new theory is self-dual. In other words, under a π rotation:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F^{\mu\nu} \\ \tilde{F}^{\mu\nu} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{F}^{\mu\nu} \\ -F^{\mu\nu} \end{pmatrix} \tag{2.1}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} j^{\nu} \\ \tilde{j}^{\nu} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{j}^{\nu} \\ -j^{\nu} \end{pmatrix} \tag{2.2}$$

and the theory remains invariant.

In fact, if we look at Maxwell's equations with both electric j^ν =(ρ_e, $\mathbf{j}_m$) and magnetic current $\tilde{j}^\nu = (\rho_m, \mathbf{j}_m)$:

$$\nabla \cdot \mathbf{E} = \rho_e \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mathbf{j}_e + \frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

$$\nabla \cdot \mathbf{B} = \rho_m \tag{2.5}$$

$$\nabla \times \mathbf{E} = -\mathbf{j}_m - \frac{\partial \mathbf{B}}{\partial t} \tag{2.6}$$

we can see this is equivalent to the well-known invariance:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{B} \\ \mathbf{E} \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{E} \\ -\mathbf{B} \end{pmatrix} \tag{2.7}$$

again, when applied with transformation (2.2).

Whilst considering the latter case, Dirac added a static point magnetic charge with coupling g at the origin, such that equation (2.5) for $\rho_m = 4\pi g \delta^{(3)}(\mathbf{r})$ reads:

$$\vec{\nabla} \cdot \mathbf{B} = 4\pi g \delta^{(3)}(\mathbf{r}) \tag{2.8}$$

which is solved by a magnetic field:

$$\mathbf{B} \;=\; g\frac{\mathbf{r}}{r^3} \tag{2.9}$$

with magnetic flux:

$$\Phi = 4\pi g \tag{2.10}$$

But what kind of smooth vector potential $\mathbf{A}$, such that $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$, generates this magnetic field? After all, from vector calculus we know that $\vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{A}) = 0$.

Dirac's key to solving this[1] was to consider two scenarios; one of infinitesimally small magnetic dipoles paired vertically, north-to-south, from the origin all the way along the positive z-axis to $+\infty$, and another one that instead goes along the negative z-axis to $-\infty$. In spherical coordinates, the former gives the solution:

$$A_r^S = A_\theta^S = 0,\ A_\phi^S = -\frac{g}{r}\frac{1+\cos\theta}{\sin\theta} \tag{2.11}$$

where we find a singularity along the positive z-axis of magnetic dipoles, called the Dirac string singularity, and the latter gives the solution:

$$A_r^N = A_\theta^N = 0,\ A_\phi^N = \frac{g}{r}\frac{1-\cos\theta}{\sin\theta} \tag{2.12}$$

with the Dirac string singularity along the negative z-axis.

It is important to note that although the singularity at the origin is physical, the rest of the Dirac string corresponds to an unphysical, or coordinate singularity. The latter can be worked around by selectively using our two solutions in two different coordinate patches. A^N is used over all points excluding the negative z-axis, and A^S is used over all points excluding the positive z-axis. That way, for each solution we avoid their respective Dirac string.

Because we assume that both solutions describe the same physical field, at the points where the two charts intersect (that is, all points excluding the z-axis) they must be related by a gauge transformation. This is:

$$A_\phi^S = A_\phi^N - \frac{2g}{r\sin\theta} \tag{2.13}$$

which, due to the general form of all gauge transformations, can be re-written as:

$$A_\phi^S = A_\phi^N - \frac{i}{e} S\nabla_\phi S^{-1} \tag{2.14}$$

for

$$S = \exp(2ige\phi) \tag{2.15}$$

Due to the nature of the spherical coordinate system, $\phi \sim \phi + 2\pi$ [1]. Therefore, the gauge transformation should be single-valued over this interval as well. As such, $S = \exp\big(2ige\phi\big) = \exp\big(2ige(\phi + 2\pi\mathbb{Z})\big)$ we find:

$$2ge = \mathbb{Z} \tag{2.16}$$

This is called the Dirac quantization condition, and it shows that if we have a single magnetic monopole of charge g in the universe, then all electric charge must be quantized as:

$$e = \frac{n}{2g} \qquad \text{for } n \in \mathbb{Z} \tag{2.17}$$

We can illuminate further what has just happened by looking at the Dirac monopole from the fibre bundle formulation. In it, we obtain electromagnetism by finding the gauge potential as a connection over a U(1)-bundle with 4-dimensional Minkowski spacetime as base space.

But what happens if we change the topology of the base space? Ignoring the time coordinate for simplicity, we remove the origin of $\mathbb{R}^3$ to obtain a base space which belongs to the same homotopy class as S^2. Now, the connection of the U(1)-bundle over S^2 requires two local charts to describe it. These can be our previous A^N and A^S over their respective north and south charts. The transition function over their overlap is the Abelian gauge transformation $A \to A + SdS^{-1}$, which brings us to equation (2.14). Therefore, enforcing this to be single-valued over the range of ϕ gives us the quantization condition [2].

It is interesting to note that instead of altering the Bianchi identity, as Dirac did via the introduction of a Dirac string, we obtain the same solution by changing the topology of the base space instead!

[1] $\sim$ refers to an equivalence relationship. i.e. ϕ is cyclic over 2π.

[2] To read about the fibre bundle formalism and the Dirac monopole, see [7]

There is a second way to derive the Dirac quantization condition that is more in line with his original paper[1], and the calculations in this dissertation.

We begin with the potential of a magnetic monopole at the origin, $\mathbf{A}(\mathbf{r})$. The wavefunction of a particle in this potential is found by making the substitution $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ to the solution for the Schrödinger equation of a free particle. This corresponds to a change in phase:

$$\langle \mathbf{r}|\psi\rangle \to e^{-ie\mathbf{A}\cdot\mathbf{r}}\langle \mathbf{r}|\psi\rangle \tag{2.18}$$

The total phase change after going in a circle around the origin at constant θ is

$$\Delta\phi = e\oint \mathbf{A}\cdot d\mathbf{r} \tag{2.19}$$

$$= e\int (\vec{\nabla}\times\mathbf{A})\cdot d\mathbf{S} \tag{2.20}$$

$$= e\int \mathbf{B}\cdot d\mathbf{S} \tag{2.21}$$

$$= e\Phi(\theta) \tag{2.22}$$

where $\Phi(\theta)$ is the flux going through the cap with points at constant distance from the origin, $0 < \phi \leq 2\pi$, and polar angle that goes from zero to θ.

As $\theta \to \pi$, $\Delta\alpha \to 0$. But since $\Phi(\pi) = 4\pi g$, $\mathbf{A}$ is singular - our Dirac string again. The Dirac veto states that the wavefunction is zero there, such that the discrepancy in phase change is not a problem. However, it must be single-valued, such that $\Delta\alpha = 2\pi n$ for $n \in \mathbb{Z}$. Therefore, $e\Phi(\pi) = 2\pi n$ and our quantization condition is:

$$eg = \frac{n}{2} \tag{2.23}$$

2.2 Solitons

A soliton is a stable solution to the equations of motion of a field theory which has non-zero energy. In the context of gauge theories, we can obtain them by changing the vacuum configuration of the fields at infinity.

A simple example[8] is the sine-Gordon kink. It is equivalent to imagining an infinite string of vertical pegs, each one attached to the next from the top and bottom, being acted on by gravity. Clearly, if all the pegs are vertically aligned, then that corresponds to a stable vacuum solution. In this scenario, the boundary at infinity corresponds to the pegs infinitely far to the right and left. We can change the boundary condition by turning by 180 degrees the peg at the boundary on the right. In this case, the pegs in between would have to turn, in order to smoothly connect the orientation of the two boundaries. Thus, each boundary condition corresponds to a different solution of the whole field, with non-zero energy.

Extending this case to a scalar field in three-dimensional spacetime, with local $U(1)$ gauge invariance, we have the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi^* D^\mu\phi - V(\phi) \tag{2.24}$$

for $V(\phi) = (a^2 - \phi^*\phi)^2$, constant a, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$.

In polar coordinates, the potential has equilibria at $\phi = ae^{in\theta}$, where $n \in \mathbb{Z}$. Since the vacuum manifold in field space is S^1, we include the phase to reflect the fact that as long as $|\phi| = a$, we have the freedom to choose a direction in field space. The integer in the phase ensure that the field in single-valued as $0 < \theta < 2\pi$, and it defines different vacuum solutions.

If we impose that at the boundary the vector potential is in the pure gauge:

$$\mathbf{A} \overset{r\to\infty}{=} \frac{1}{e}\nabla(n\theta) \tag{2.25}$$

then $F_{\mu\nu} = 0$, $D_\mu\phi = 0$, and it corresponds to a vacuum solution. Now, assuming that at the boundary the field is also in the previously mentioned vacuum solution, we find:

$$\mathcal{H} = -\mathcal{L} \quad \text{(for a static configuration)} \tag{2.26}$$

and therefore $\mathcal{H} \to 0$ as $r \to \infty$, and the solution has finite energy over the whole space.

In exchange for obtaining a stable, finite energy field configuration, we introduced a vector potential with non-zero flux:

$$\Phi = \oint \mathbf{A} \cdot d\mathbf{l} = \int A_\theta r d\theta = -\frac{2\pi n}{e} \tag{2.27}$$

which is quantized!

In summary, for each value of n, we have a field configuration at infinity which cannot be smoothly deformed via a gauge transformation to any of the other ones. This configuration has finite energy, is stable, and due to the $U(1)$ potential has a non-zero electromagnetic field with quantized flux! This will be explained in more detail in the next section.

2.3 The 't Hooft-Polyakov monopole

In this case, we consider a gauge theory with a scalar field in the fundamental representation of O(3):

$$\mathcal{L} = -\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - \frac{1}{2}m^2\phi^a\phi^a - \lambda(\phi^a\phi^a)^2 \tag{2.28}$$

where

$$D_\mu\phi^a = \partial_\mu\phi^a + e\varepsilon^{abc}A^b_\mu\phi^c \tag{2.29}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon^{abc}A^b_\mu A^c_\nu \tag{2.30}$$

and $|\phi|^2 = F^2 = -m^2/4\lambda$ defines the broken vacuum submanifold.

After spontaneous symmetry breaking (SSB), what remains is an unbroken U(1) symmetry ($SO(3) \xrightarrow{SSB} SO(2) \simeq U(1)$). Usually, the gauge with broken configuration $\phi^a = (0, 0, F)$ is chosen globally, such that we have the massless vector field $\mathcal{A}_\mu = A^3_\mu$ and field strength $\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu$.

In his famous paper 't Hooft[2] defined these quantities in a different gauge, where the field breaks radially outwards in field space, and proposed the following gauge invariant field strength:

$$\mathcal{F}_{\mu\nu} = \frac{1}{|\phi|}\phi^a F^a_{\mu\nu} - \frac{1}{e|\phi|^3}\varepsilon_{abc}\phi^a(D_\mu\phi^b)(D_\nu\phi^c) \tag{2.31}$$

such that for

$$\mathcal{A}_\mu = \frac{1}{|\phi|}\phi^a A^a_\mu \tag{2.32}$$

then

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - \frac{1}{e|\phi|^3}\varepsilon_{abc}\phi^a(\partial_\mu\phi^b)(\partial_\nu\phi^c) \tag{2.33}$$

A key point to note is that since (2.31) is gauge invariant, we correctly recover the expressions for the original gauge by simple substituting $A^3_\mu \equiv A_\mu \neq 0$ and $\phi^3 = F$ (with the rest being equal to zero).

Next, he proposed the spherically symmetric ansatz at the boundary:

$$\phi^a = Fr^a/r \tag{2.34}$$

$$A^a_i = \varepsilon^{iab}\frac{r^b}{er^2} \tag{2.35}$$

$$A^a_0 = 0 \tag{2.36}$$

which mixes spatial and gauge indices in spherical coordinates, such that at each point in the boundary the fields aim radially outwards as defined in physical space. He called this a hedgehog solution.

Substituting into our expression for the field strength we find

$$\mathcal{F}_{0i} = 0 \tag{2.37}$$

$$\mathcal{F}_{ij} = -\frac{1}{er^3}\varepsilon_{ijk}r^k \tag{2.38}$$

such that

$$B^k = \frac{r^k}{er^3} \tag{2.39}$$

which looks like the field of a magnetic monopole!

Since $\mathcal{A}_i = (1/|\phi|)\phi^a A_i^a = (1/F)(Fr^a/r)(\varepsilon^{iab}r^b/er^2) \propto \varepsilon^{iab}r^{[a}r^{b]} = 0$, it is the scalar field part of (2.31) that is responsible for this magnetic monopole, only this time the full spherically symmetric ansatz is singularity free![2] It is only at infinity that our solution looks like a magnetic monopole.

For a theory with gauge symmetry G that undergoes spontaneous symmetry breaking into another group H, ($G \xrightarrow{SSB} H$) we are interested in the gauge transformations in G which are not related by H since this is the remaining symmetry. This is precisely the coset space G/H.

Since the boundary at infinity in physical space is S^2, we define an equivalence relation for all two-spheres which are mapped onto G/H by whether they can be continuously deformed into each other. With the appropriate group action[3] this becomes the second homotopy group, and it is labelled π_2(G/H).

Being more specific, the field at the boundary in physical space is non-zero and has constant magnitude but different angles in field space, such that it defines a map from points in S^2 to the manifold G/H. If this map can be continuously deformed to a point, then the field at all points in the boundary can be smoothly deformed to the same direction in field space, and thus we don't have a soliton. Therefore, if π_2(G/H) is trivial, all field configurations in the broken state at the boundary

[3] two inequivalent loops are mapped to a different one, defined by travelling the first loop and then the second.

can be continuously deformed into the same location in coset group space, and no soliton solution exists. However, if this group is non-trivial, then soliton solutions are allowed via spontaneous symmetry breaking.

In our previous example the scalar field defined a map ϕ: $S^2 \to SO(3)/SO(2) \simeq S^2$, such that $\pi_2(S^2) = \mathbb{Z}$ (i.e. the group of integers under addition) is non-trivial. Hence, we were able to find the magnetic monopole as a topological soliton.

Although our Dirac and 't Hooft-Polyakov monopoles look very different - one has a singularity at infinity and the other doesn't, they correspond to the same solution and can be related via the appropriate gauge transformation.

To do this, we first note that our previous result for the 't Hooft-Polyakov monopole is equivalent to a scalar field in the adjoint representation of SU(2), which breaks into U(1). Since these are equivalent representations.

We begin by embedding our Dirac monopole in an SU(2) theory with adjoint scalar field in the usual broken state $\phi^3 = T^3 F$ where $\mathcal{A}_\mu = T^3 A^3_\mu$, such that:

$$A_t = A_r = A_\theta = 0 \quad \text{and} \quad A_\phi = T^3\Big(-\frac{g}{r}\Big)\Big(\frac{1-\cos\theta}{\sin\theta}\Big) \tag{2.40}$$

where we use the basis $A_\mu = A^a_\mu T^a$.

It can be shown that:

$$S = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\phi}\sin\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in SU(2) \tag{2.41}$$

such that it defines the gauge transformation

$$A_\mu \to S A_\mu S^{-1} + \frac{2i}{e} S\partial_\mu S^{-1} \tag{2.42}$$

$$\phi \to S\phi S^{-1} \tag{2.43}$$

and we have non-zero elements:

$$A_\theta = \frac{1}{er}(T^1 \sin\phi - T^2 \cos\phi) \tag{2.44}$$

$$A_\phi = \frac{1}{er}(T^1 \cos\theta \cos\phi + T^2 \cos\theta \sin\phi - T^3 \sin\theta) \tag{2.45}$$

$$\phi = F(T^1 \sin\theta \cos\phi + T^2 \sin\theta \sin\phi + T^3 \cos\theta) \tag{2.46}$$

which in Cartesian coordinates are our solutions (2.34)-(2.36)!

Thus, we can say 't Hooft-Polyakov = Dirac up to a gauge transformation, such that all "responsibility" for the magnetic monopole is moved from the scalar field to the gauge field, introducing a Dirac string singularity in the process.

2.4 Instantons

If "Euclideanise" spacetime such that $t \to it$, the boundary at infinity for the whole manifold turns to S^3. Since a soliton solution over this space would also involve boundary conditions which vary over time, this is instead called an instanton solution. There is much literature on the topic, and the reader is directed to [8] for a good introduction.

Shifting our attention to Einstein-Cartan (EC) theory with complexified time, the vector potential now defines a map from S^3 to $SO(4)$. Thus, by our previous arguments, we are interested in $\pi_3(SO(4))$, which is non-trivial, and we can have instanton solutions without the need of spontaneous symmetry breaking.

Reference [9] lays out one such solution by considering EC's equations of motion without matter:

$$e_{abcd} e^a \wedge e^b \wedge R^{cd} = 0 \tag{2.47}$$

which is solved for a (anti-) self-dual field strength, $\tilde{R}^{ab} = \frac{1}{2}\epsilon^{abcd}R_{cd} = \pm R^{ab}$ since:

$$e_{abcd}e^b \wedge R^{cd} = 2e^b \wedge \tilde{R}_{ab} \tag{2.48}$$

$$= \pm 2R_{ab} \wedge e^b \tag{2.49}$$

$$= \pm 2DT_a \tag{2.50}$$

$$= 0 \tag{2.51}$$

where $DT^a = 0$ is the cyclic identity of the Riemann tensor for zero torsion.

We simplify this further by noticing that since

$$\tilde{R} = d\tilde{\omega} + \frac{1}{2}\big(\tilde{\omega} \wedge \omega + \omega \wedge \tilde{\omega}\big) \tag{2.52}$$

an (anti-) self-dual spin connection leads to an (anti-) self-dual curvature, and thus we have an instanton solution for an equation of motion which is first order in the tetrad field.

Chapter 3

Linearized gravity

3.1 Dirac monopole... again!

In 1985 Zee A. introduced[3] to linearized Einstein gravity, an analogue of the Dirac monopole. Motivated by the well known similarity between Einstein's equations in this regime and Maxwell's equations of electrodynamics, he speculated on whether there could be physical examples of magnetic monopoles - as Dirac predicted them for electromagnetism - in gravity, and what the consequences of this would be.

By perturbing the metric from flat Minkowski spacetime as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we can define the potentials $\phi = h_{00}$ and $\zeta^i = h_{0i}$, and therefore the spatial 1-forms $\mathbf{g} = -\vec{\nabla}\phi$ and $\mathbf{B} = -\vec{\nabla} \times \zeta$.

If we then substitute these into Einstein's field equations, and choose what appears to be the Lorentz gauge $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0 \Rightarrow 4\partial\phi/\partial t + \vec{\nabla} \cdot \zeta = 0$, we obtain:

$$\vec{\nabla} \cdot \mathbf{g} = -4\pi G\rho \tag{3.1}$$

$$\vec{\nabla} \times \mathbf{B} = -16\pi G\mathbf{K} + \frac{\partial \mathbf{g}}{\partial t} \tag{3.2}$$

$$\vec{\nabla} \times \mathbf{g} = 0 \tag{3.3}$$

$$\vec{\nabla} \cdot \mathbf{B} = 0 \tag{3.4}$$

which look like Maxwell's equations, for ρ and $\mathbf{K}$ the lowest order[1] perturbations of the energy-momentum tensor components T_{00} and T_{0i} respectively. These indeed correspond to our usual (we will get back to this later) notions of energy density and momentum density respectively for the system.

Like for EM, equation (3.4) follows from the fact that for a smooth vector potential $\vec{\nabla} \cdot (\vec{\nabla} \times \zeta) = 0$ is inevitable. Thus, Zee proceeds along the lines of Dirac and introduces the term:

$$\vec{\nabla} \cdot \mathbf{B} = 4\pi\gamma\delta^{(3)}(r) \tag{3.5}$$

[2]

such that γ corresponds to the magnetic/dual mass (as opposed to the usual "electric" mass) of what we now label as a gravitational magnetic monopole or gravipole.

Considering the action for a point particle in this perturbed metric:

$$S = -m \int d\tau \tag{3.6}$$

$$= \int dt \Big(-m + m\Big[\frac{1}{2}\mathbf{v}^2 + \frac{1}{8}(\mathbf{v}^2)^2\Big] - m\Big[\phi + \frac{1}{2}\phi^2 + \psi + \frac{3}{2}\phi v^2\Big] - m\boldsymbol{\zeta} \cdot \mathbf{v}\Big) \tag{3.7}$$

[3]

The first and second terms are the relativistically corrected kinetic and potential energy, respectively. The last term, which can be written as the line integral:

$$-m \int d\mathbf{x} \cdot \boldsymbol{\zeta} \tag{3.8}$$

is the one of interest, since for a particle orbiting far enough away such that this post-Newtonian approximation still applies, it is the total phase change in the wavefunction. Therefore, we can apply -exactly- the same arguments as in the last part

[1] of parameter $\sim (GM/r)^{1/2}$ which becomes smaller for the weak gravity limit

[2] Although not in the original paper by Zee, the writer of this dissertation included 4π to make the connection with the end of sec. 2.1 clear.

[3] $\psi = -g_{00}^{(4)}/2 - \phi^2$ where $g_{00}^{(4)}$ is next order correction to h_{00}.

of sec. 2.1 and conclude the analogous mass quantization condition:

$$m\gamma = \frac{\mathbb{Z}}{2} \tag{3.9}$$

Moving on to the equations of motion of (3.7), we obtain an analogue of the Lorentz law. A new force points away from the plane of rotation such that were there to be a gravipole inside the sun, the orbital plane of the planet would no longer intersect the central star - ideally something we could measure for our own sun!

This equation of motion is:

$$\frac{d\mathbf{v}}{dt} = -\frac{GM}{r^2}\hat{\mathbf{r}} + \mathbf{v} \times \frac{\gamma}{r^2}\hat{\mathbf{r}} \tag{3.10}$$

Other speculations that Zee raises are:

1. If instead of a particle, we consider moving a given nucleus around a gravipole, then our quantization condition would apply to every possible nucleus, thus leading to the quantization of all possible binding energies inside the nucleus. This, in turn, would lead to some kind of constraints on the fundamental couplings of nature. (At least for the strong and electromagnetic couplings.)

2. If mass is quantized, then so is photon energy. Although our post-Newtonian approximation wouldn't hold for the orbit of a massless particle, we can consider the following thought experiment - place a photon inside a stationary box. The increase in energy of the box is equivalent to an increase in its mass, which can only be increased in multiples of its smallest quantized value. However, this energy comes purely from the photon, which therefore must

also increase only in quantized amounts.

3. Friedman and Sorkin's[10] topological solution of spacetime contains spin-1/2 particle solutions. This could be motivation that in some modified/generalized theory of gravity, the gravipole might exist as a topological solution.

4. Montonen and Olive[11] proposed that in a quantized theory, the magnetic monopole solutions form a triplet with the photon - perhaps, the gravipole and graviton similarly form a representation under some dual group.[4]

3.2 Spin-2 duality

In section 2.1, we saw that by adding a magnetic current to the Bianchi identity, a new duality invariance of the theory appears - equations of motion remain the same after a π rotation of the field strength and its dual. Paper [13] extends this to linearized gravity by rotating the Riemann tensor and its dual into each other:

$$R'_{\lambda\mu\rho\sigma} = \cos\alpha\ R_{\lambda\mu\rho\sigma} + \sin\alpha\ S_{\lambda\mu\rho\sigma} \tag{3.11}$$

$$S'_{\lambda\mu\rho\sigma} = -\sin\alpha\ R_{\lambda\mu\rho\sigma} + \cos\alpha\ S_{\lambda\mu\rho\sigma} \tag{3.12}$$

such that not only the equations of motion remain invariant, but also the action. Therefore, defining an $SO(2)$ invariance of the theory.[5]

Starting from this principle, [14] adds symmetric electric and magnetic sources to linearized spin-2 theory, and develops a general equation for its quantization condition, the derivation of which we will cover in this section and the next.

[4]In [12] he suggests that a possible solution to the cosmological constant problem could be to change the dimensionality of the constant term of the Lagrangian, as was done for an interaction term in the proton decay problem, by making the graviton some kind of composite particle of a higher theory.

[5]The equations of motion are invariant under any $GL(2,\mathbb{R})$ transformation, however only a rotation will leave the action invariant.

These electric and magnetic sources, $T_{\mu\nu}$ and $\Theta_{\mu\nu}$ respectively, are conserved such that $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu \Theta^{\mu\nu} = 0$. Since in our earlier example, the current and its dual also had to transform under the $SO(2)$ rotation, in this case we have the rotation of our energy-momentum tensors:

$$T'_{\alpha\beta} = \cos\alpha \ T_{\alpha\beta} + \sin\alpha \ \Theta_{\alpha\beta} \tag{3.13}$$

$$\Theta'_{\alpha\beta} = -\sin\alpha \ T_{\alpha\beta} + \cos\alpha \ \Theta_{\alpha\beta} \tag{3.14}$$

Under this duality, the relations below hold:

$$R_{\alpha\beta\lambda\mu} = R_{[\alpha\beta][\lambda\mu]} \tag{3.15}$$

For the trace reversed $\bar{\Theta}_{\alpha\beta} = \Theta_{\alpha\beta} - (1/2)\eta_{\alpha\beta}\Theta$, where $\Theta = \eta_{\alpha\beta}\Theta^{\alpha\beta}$:

$$R_{\alpha\beta\lambda\mu} + R_{\lambda\alpha\beta\mu} + R_{\beta\lambda\alpha\mu} = 8\pi G \epsilon_{\alpha\beta\lambda\nu} \bar{\Theta}^{\nu}{}_{\mu} \tag{3.16}$$

A very interesting point to notice that the paper doesn't reference, is the fact that this corresponds to breaking the cyclic symmetry of the Riemann tensor in GR; $DT = R \wedge e$, which can only happen for non-zero torsion! - This raises whether by adding a magnetic monopole, we immediately add torsion to the system.

By permuting the indices, this leads to:

$$R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} = 4\pi G(\epsilon_{\alpha\beta\gamma\lambda} \ \bar{\Theta}^{\lambda}{}_{\delta} - \epsilon_{\alpha\beta\lambda\lambda} \ \bar{\Theta}^{\lambda}{}_{\gamma} + \epsilon_{\beta\gamma\delta\lambda} \ \bar{\Theta}^{\lambda}{}_{\alpha} - \epsilon_{\alpha\gamma\delta\lambda} \ \bar{\Theta}^{\lambda}{}_{\beta}) \tag{3.17}$$

The Bianchi identity:

$$\partial_\epsilon R_{\alpha\beta\gamma\delta} + \partial_\alpha R_{\beta\epsilon\gamma\delta} + \partial_\beta R_{\epsilon\alpha\gamma\delta} = 8\pi G \epsilon_{\epsilon\alpha\beta\rho}(\partial_\gamma \bar{\Theta}^{\rho}{}_{\delta} - \partial_\delta \bar{\Theta}^{\rho}{}_{\gamma}) \tag{3.18}$$

which reinforces the electric energy-momentum tensor conservation via:

$$\partial_\mu G^{\mu\nu} = 0 \tag{3.19}$$

And the Einstein field equations:

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} \tag{3.20}$$

These equations are now symmetric under our duality transformations. As a result, we can obtain the same equations for the dual Riemann with the electric and magnetic energy-momentum tensors rotated. And in the absence of any sources, we recover the usual symmetries of the Riemann tensor and equations for linearized gravity.

Next, we need to introduce an action in order to obtain our equations of motion. Having said that, in order to invoke the variational principle, we need to introduce the usual spin-2 field $h_{\mu\nu} = h_{\nu\mu}$.

In a scenario without sources (i.e. $T_{\mu\nu} = \bar{\Theta}_{\mu\nu} = 0$) the two identities (3.16) and (3.18) imply the existence of our familiar symmetric tensor gauge field, such that:

$$R_{\mu\nu\lambda\rho} = \partial_{[\lambda} h_{\mu][\sigma,\rho]} \tag{3.21}$$

However, when we introduce our corresponding energy-momentum tensors these symmetries are broken. To proceed, we introduce two components that make up the Riemann tensor; one that satisfies the vacuum symmetries and one that is fixed by the magnetic energy-momentum tensor. The latter is defined by:

$$\partial_\alpha \Phi^{\alpha\beta}{}_{\gamma} = 16\pi G\, \Theta^{\beta}{}_{\gamma} \tag{3.22}$$

where $\Phi^{\alpha\beta}{}_{\gamma} = \Phi^{[\alpha\beta]}{}_{\gamma}$ since the magnetic energy-momentum is conserved; $\partial_\beta \Theta^{\beta}{}_{\gamma} = \frac{1}{16\pi G} \partial_{[\beta}\partial_{\alpha]} \Phi^{[\alpha\beta]}{}_{\gamma} = 0$.

With this, we set:

$$R_{\lambda\mu\alpha\beta} = r_{\lambda\mu\alpha\beta} + \frac{1}{4}\epsilon_{\lambda\mu\rho\sigma}\left(\partial_\alpha \bar{\Phi}^{\rho\sigma}{}_{\beta} - \partial_\beta \bar{\Phi}^{\rho\sigma}{}_{\alpha}\right) \tag{3.23}$$

where $\bar{\Phi}^{\alpha\beta}{}_{\gamma} = \Phi^{\alpha\beta}{}_{\gamma} + \frac{1}{2}\Big(\delta^{\alpha}_{\gamma}\Phi^{\beta} - \delta^{\beta}_{\gamma}\Phi^{\alpha}\Big)$ and $\Phi^{\alpha} \equiv \Phi^{\alpha\sigma}{}_{\sigma}$.

Given this, it is straight forward to check that

$$\bar{\Phi}^{\alpha} = -\frac{1}{2}\Phi^{\alpha} \tag{3.24}$$

and

$$\partial_{\alpha}\bar{\Phi}^{\alpha\beta}{}_{\gamma} = 16\pi G\ \bar{\Theta}^{\beta}{}_{\gamma} - \partial_{\gamma}\bar{\Phi}^{\beta} \tag{3.25}$$

such that $r_{\alpha\beta\gamma\delta}$ satisfies the cyclic and Bianchi identities respectively:

$$r_{\alpha\beta\gamma\delta} + r_{\gamma\alpha\beta\delta} + r_{\beta\gamma\alpha\delta} = 0, \qquad \partial_{\epsilon}r_{\alpha\beta\gamma\delta} + \partial_{\alpha}r_{\beta\epsilon\gamma\delta} + \partial_{\beta}r_{\epsilon\alpha\gamma\delta} = 0 \tag{3.26}$$

Now we can define the usual symmetric tensor by: $r_{\alpha\beta\gamma\delta} = \partial_{[\alpha}h_{\beta][\gamma,\delta]}$ and with $y^{\alpha\beta}{}_{\lambda} = \epsilon^{\alpha\beta\gamma\delta}\partial_{\gamma}h_{\delta\lambda}$ rewrite the curvature as:

$$R_{\lambda\mu\alpha\beta} = \frac{1}{4}\epsilon_{\lambda\mu\rho\sigma}\Big(\partial_{\alpha}\bar{Y}^{\rho\sigma}{}_{\beta} - \partial_{\beta}\bar{Y}^{\rho\sigma}{}_{\alpha}\Big) \tag{3.27}$$

for

$$Y^{\rho\sigma}{}_{\beta} = y^{\rho\sigma}{}_{\beta} + \Phi^{\rho\sigma}{}_{\beta}, \quad \bar{Y}^{\rho\sigma}{}_{\alpha} = Y^{\rho\sigma}{}_{\alpha} + \frac{1}{2}\Big(\delta^{\rho}_{\alpha}Y^{\sigma} - \delta^{\sigma}_{\alpha}Y^{\rho}\Big), \quad Y^{\rho} \equiv Y^{\rho\sigma}{}_{\sigma} \tag{3.28}$$

Next, we introduce explicit expressions for our electric and magnetic energy-momentum tensors for a point source:

$$T^{\mu\nu} = \frac{u^{\mu}u^{\nu}}{u^{0}}\delta^{(3)}(\mathbf{x} - \mathbf{z}(x^{0})), \qquad \Theta^{\mu\nu} = \frac{v^{\mu}v^{\nu}}{v^{0}}\delta^{(3)}(\mathbf{x} - \bar{\mathbf{z}}(x^{0})) \tag{3.29}$$

where z^{μ} and $\bar{z}^{\mu}$ are the world lines of the electric and magnetic charge respectively, $u^{\mu} = \frac{dz^{\mu}}{d\lambda}$ and $v^{\mu} = \frac{d\bar{z}^{\mu}}{d\lambda}$.

Given this, to solve equation (3.22) a Dirac string is introduced with coordinates $y^{\alpha}(\lambda,\sigma)$. (λ,σ) parametrize its worldsheet, and since the string always starts at the point magnetic source, $(\lambda,\sigma = 0)$ parametrizes its worldline.

The solution is:

$$\Phi^{\alpha\beta}{}_{\gamma} = 16\pi G N v_{\gamma} \int d\lambda d\sigma (y'^{\alpha}\dot{y}^{\beta} - y'^{\beta}\dot{y}^{\alpha})\delta^{(4)}(x - y(\lambda,\sigma)) \tag{3.30}$$

where $\dot{y}^{\alpha} = \frac{\partial y^{\alpha}}{\partial \lambda}$, $y'^{\alpha} = \frac{\partial y^{\alpha}}{\partial \sigma}$, N is the magnetic mass and the conserved magnetic 4-momentum $N v_{\gamma}$ plays the role of the magnetic coupling.

Finally, the action from which (3.20) can be obtained is:

$$S[h_{\mu\nu}(x), y^{\rho}(\lambda,\sigma)] = \frac{1}{16\pi G}\int \frac{1}{4}(\bar{Y}_{\alpha\beta\gamma}\bar{Y}^{\alpha\gamma\beta} - \bar{Y}_{\alpha}\bar{Y}^{\alpha})d^4x + \frac{1}{2}\int h_{\mu\nu}T^{\mu\nu}d^4x \tag{3.31}$$

Interestingly, we vary the field $h_{\mu\nu}$ and the Dirac string y^{μ} (as long as it remains attached to the magnetic source), but we don't vary the electric and magnetic sources, which is a limitation of the linearized theory in general even just with electric sources.

For no magnetic source, this action reduces to:

$$S^{PF} = \frac{1}{16\pi G}\int \frac{1}{4}\Big(- \partial_{\lambda}h_{\alpha\beta}\partial^{\lambda}h^{\alpha\beta} + 2\partial_{\lambda}h^{\lambda\alpha}\partial^{\mu}h_{\mu\alpha} - 2\partial^{\lambda}h\partial_{\mu}h^{\mu\lambda} + \partial_{\lambda}h\partial^{\lambda}h\Big) \tag{3.32}$$

which is called the Pauli-Fierz action, which describes a spin-2 gauge field as previously mentioned, and if we vary it by $h_{\alpha\beta}$ it gives Einstein's linearized field equations.

Due to the freedom to choose a coordinate system, the action (3.31) must be diffeomorphism invariant, as well as invariant under translations of the Dirac string (with an appropriate transformation of our spin-2 field) such that our Dirac string remains classically unobservable.

If $Y^{\mu\nu}{}_{\alpha}$ transforms in the form:

$$\bar{Y}^{\mu\nu}{}_{\alpha} \to \bar{Y}^{\mu\nu}{}_{\alpha} + \partial_{\alpha}z^{\mu\nu} \tag{3.33}$$

where $z^{\mu\nu} = z^{[\mu\nu]}$, the Riemann tensor remains invariant, as does the first element of (3.31) up to a boundary term.

Now considering an infinitesimal diffeomorphism,

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{3.34}$$

then

$$Y^{\mu\nu}{}_\gamma \to Y^{\mu\nu}{}_\gamma + \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\gamma\xi_\beta \tag{3.35}$$

and

$$\bar{Y}^{\mu\nu}{}_\gamma \to \bar{Y}^{\mu\nu}{}_\gamma + \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\partial_\gamma\xi_\beta \tag{3.36}$$

Thus we can write $z^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha\xi_\beta$ and the first element of our action is indeed invariant. The second, minimally coupled term is invariant (up to a boundary term) because $T_{\mu\nu}$ is conserved.

For a displacement of the Dirac string:

$$y^\alpha(\lambda,\sigma) \to y^\alpha(\lambda,\sigma) + \delta y^\alpha(\lambda,\sigma) \tag{3.37}$$

$$\Phi^{\mu\nu}{}_\alpha \to \Phi^{\mu\nu}{}_\alpha + k^{\mu\nu}{}_\alpha \tag{3.38}$$

For the last, we don't need to find the exact form of $k^{\mu\nu}{}_\alpha$ from (3.30), as long as $\partial_\mu k^{\mu\nu}{}_\alpha = 0$. Which must be the case since the magnetic energy-momentum tensor is invariant under a displacement of the Dirac string (recall equation 3.22).

If we also apply a general diffeomorphism:

$$Y^{\mu\nu}{}_\alpha \to Y^{\mu\nu}{}_\alpha + \epsilon^{\mu\nu\rho\sigma}\partial_\rho\delta h_{\sigma\alpha} + k^{\mu\nu}{}_\alpha \tag{3.39}$$

For our previous condition, we can write $k^{\mu\nu}{}_\alpha = \partial_\rho t^{\mu\nu\rho}{}_\alpha$, where $t^{\mu\nu\rho}{}_\alpha = t^{[\mu\nu\rho]}{}_\alpha$ such that $\partial_\mu k^{\mu\nu}{}_\alpha = 0$. This can be further simplified by $t^{\mu\nu\rho}{}_\alpha = \epsilon^{\mu\nu\rho\sigma}(a_{\sigma\alpha} + s_{\sigma\alpha})$, where $a_{\sigma\alpha} = a_{[\sigma\alpha]}$ and $s_{\sigma\alpha} = s_{(\sigma\alpha)}$. Choosing the appropriate diffeomorphism, $\delta h_{\mu\nu} = -s_{\mu\nu}$, then

$$Y^{\mu\nu}{}_{\alpha} \to Y^{\mu\nu}{}_{\alpha} + \epsilon^{\mu\nu\rho\sigma}\partial_{\rho}a_{\sigma\alpha} \tag{3.40}$$

such that for $z_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}a^{\rho\sigma}$, we obtain (3.33).

An important point is that the spin-2 field is only varied for $k^{\mu\nu}{}_{\alpha} \neq 0$. However, y^{α} has support (the spacetime domain over which we vary) in the string locations, which can't intersect the electric mass worldline due to the Dirac veto. The consequence of this is that the minimally coupled term in the action also remains invariant (i.e. if $\delta h_{\mu\nu} \neq 0$ then $T_{\mu\nu} = 0$ and vice versa).

Thus, the first two elements of (3.31) remain invariant -as well as the Riemann tensor- under translations of the Dirac string.

3.3 Quantization condition

With the linearized spin-2 theory set-up with magnetic and electric sources, [14] describes the process of quantizing the system by following a recipe from Dirac[15] to make the Dirac string quantum mechanically unobservable - which inevitably leads to the charge quantization condition.

So far we've established that the Dirac string is classically unobservable due to the invariance of the action under perturbations of the string. This induces first class constraints (in a gauge $y^0 = \lambda$) with conjugate momentum for each spatial coordinate (y^m)

$$\pi_m = \frac{\partial \mathcal{L}}{\partial \dot{y}^m} \tag{3.41}$$

$$= \frac{\partial \mathcal{L}}{\partial Y^{\alpha\beta}_{\ \ \gamma}} \frac{\partial Y^{\alpha\beta}_{\ \ \gamma}}{\partial \dot{y}^m} \tag{3.42}$$

$$= \frac{\partial \mathcal{L}}{\partial Y^{\alpha\beta}_{\ \ \gamma}} \frac{\partial \Phi^{\alpha\beta}_{\ \ \gamma}}{\partial \dot{y}^m} \tag{3.43}$$

$$= 32\pi G N v_\gamma \frac{\partial \mathcal{L}}{\partial Y^{\alpha\beta}_{\ \ \gamma}} \int d\lambda d\sigma y'^{[\alpha} \delta^{\beta]}_m \delta^{(4)}(x - y(\lambda, \sigma)) \tag{3.44}$$

$$= -\, 32\pi G N y'^n v_\gamma \frac{\partial \mathcal{L}}{\partial Y^{mn}_{\ \ \ \gamma}} \tag{3.45}$$

where we used the fact that $Y^{mn}_{\ \ \ \gamma} = Y^{[mn]}_{\ \ \ \ \gamma}$ and assumed that x intersects a Dirac string (otherwise it vanishes). Since this generates the change in the gravitational field after a shift in the Dirac string, by varying the wave functional with respect to the string location in the quantum theory we obtain:

$$\frac{1}{i} \frac{\delta \Psi}{\delta y^m(\sigma)} = -32\pi G N y'^n v_\gamma \frac{\partial \mathcal{L}}{\partial Y^{mn}_{\ \ \ \gamma}} \Psi \tag{3.46}$$

such that if $\delta\Psi = \alpha\Psi$ for small enough alpha,

$$\Psi \to \Psi + \delta\Psi \tag{3.47}$$

$$= \Psi + \alpha\Psi \tag{3.48}$$

$$= (1 + \alpha)\Psi \tag{3.49}$$

$$\approx e^{\alpha}\Psi \tag{3.50}$$

and the total phase change in the wave functional, after the string sweeps a two-dimensional closed surface around an electric pole is:

$$\Delta\Psi = -16 G N v_\gamma \int \frac{\partial \mathcal{L}}{\partial Y^{mn}_{\ \ \ \gamma}} (\dot{y}^m y'^n - \dot{y}^n y'^m) d\lambda d\sigma \tag{3.51}$$

$$= -16 G N v_\gamma \int d^3x \epsilon^{mnp} \partial_p \Big(\frac{\partial \mathcal{L}}{\partial Y^{mn}_{\ \ \ \gamma}} \Big) \tag{3.52}$$

where we used Gauss's theorem on the last line.

Considering the first term in (3.31) -call this $S_1 = \int \mathcal{L}_1$- we know

$$
\begin{aligned}
-\frac{1}{16\pi G}G^{\alpha\beta} &= \frac{\delta S_1}{\delta h_{\alpha\beta}} &(3.53)\\
&= -\partial_\rho\left(\frac{\partial \mathcal{L}_1}{\partial Y^{\mu\nu}{}_\sigma}\frac{\partial Y^{\mu\nu}{}_\sigma}{\partial\partial_\rho h_{\alpha\beta}}\right) &(3.54)\\
&= -\partial_\rho\left(\frac{\partial \mathcal{L}_1}{\partial Y^{\mu\nu}{}_\beta}\right)\epsilon^{\mu\nu\rho\alpha} &(3.55)\\
&= -\partial_\rho\left(\frac{\partial \mathcal{L}_1}{\partial Y^{\mu\nu}{}_\alpha}\right)\epsilon^{\mu\nu\rho\beta} \quad \text{(A property[6] of } \mathcal{L}_1 \text{ due to gauge invariance)} &(3.56)
\end{aligned}
$$

and substituting this into (3.52) we find

$$\Delta\Psi = 8\pi G N v_\gamma \int d^3x T^{0\gamma} = 8\pi G N M v_\gamma u^\gamma \tag{3.57}$$

As usual, by imposing that the wavefunction is single-valued we obtain:

$$4GNMv_\gamma u^\gamma = n, \quad n \in \mathbb{Z} \tag{3.58}$$

This is actually a relation for the 4-momenta of the poles:

$$4GP_\gamma Q^\gamma \in \mathbb{Z} \tag{3.59}$$

for $P_\gamma = Mu_\gamma$ and $Q_\gamma = Nv_\gamma$.

In fact, if we choose the centre of momentum frame for the magnetic mass, we obtain the relation:

$$4GEN \in \mathbb{Z} \tag{3.60}$$

and thus our quantization condition is for the energy, not the mass! Thus demonstrating in more generality (within the linearized limit) that Zee's speculation on

[6] By inspection it also imposes the contracted Bianchi identities.

section 3.1 on quantization of energy rather than mass is more accurate.

Had we considered two dyons - poles with both electric and magnetic mass - each with charges (P^γ,Q^γ) and ($\bar{P}^\gamma, \bar{Q}^\gamma$) respectively, our quantization condition would read:

$$4G\big(P_\gamma \bar{Q}^\gamma - \bar{P}_\gamma Q^\gamma\big) \equiv 4G\epsilon_{ab}Q^a_\gamma \bar{Q}^{b\gamma} \in \mathbb{Z} \tag{3.61}$$

where the last expression is explicitly duality rotation invariant, since ϵ^{ab} is the $SO(2)$-invariant Levi-Civita tensor, for the space with indices that determine charge and dual charge.

Chapter 4

Taub-NUT

So far, in our quest for gravipoles we have looked at the concept of manually adding them for linearized gravity, with the hopes that a higher theory will decompose to this general solution. However, maybe there is no need for this, since there already exists a solution to the full (non-linear) Einstein field equations which returns the Zee gravipole! It is called the Taub-NUT spacetime.

4.1 Classical solution

In 1950 A. H. Taub[16] discovered the time-dependent part of the whole spacetime, and in 1963, Newman, Tamburino and Unti[17] rediscovered in as a simple generalization of the Schwarzschild spacetime, with coordinates that cover the whole manifold: both the stationary and time-dependent regions.

The Taub-NUT solution to the Einstein field equations is:

$$ds^2 = -f(r)(dt + 2l(k - \cos\theta)d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.1}$$

where

$$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2} \tag{4.2}$$

for $k = 0$, with parameters[1] m and l.

The first parameter becomes the mass of the source in the Schwarzschild limit, and it is generally accepted to be the electric mass (check out chapter 3 for terminology), a notion that is nicely supported by the fact that it is always positive, since its sign can be reversed by the coordinate transformation $r \to -r$.

In the context of gravity monopoles, the second is called the NUT parameter, and it is interpreted to be the dual/magnetic mass. If we let $l = 0$ and $m \neq 0$ our solution reduces to the Schwarzschild metric, and [18] uses the Weyl curvature of Taub-NUT to show that for non-zero l geodesics will twist, although the possible sources of the twist need to be investigated.

Shifting the constant k by $k \to k + \alpha$ is equivalent to doing the coordinate transformation $t \to t + 2l\alpha\phi$[2].

From now on, we take the generally adopted form, with $k = 1$:

$$ds^2 = -f(r)(dt + 4l\sin^2\frac{1}{2}\theta d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.3}$$

and it has a string-like singularity on the negative z-axis, for $\theta = \pi$ (for other values of k the singularity changes location). This singularity is present for $l \neq 0$. Otherwise, our solution becomes Schwarzschild, which doesn't have this kind of divergence.

The spacetime is asymptotically flat in the sense that as $r \to \infty$, the Riemann tensor decays as r^{-3}. However, since there is a singularity at $\theta = \pi$, for $l \neq 0$ the spacetime cannot be globally asymptotically flat.

[1] This solution also contains another parameter; $\epsilon = +1$. It is what we refer to as the Taub-NUT solution. This is also the only case which includes the Schwarzschild solution[18].

[2] [14] has a typo - equation (IV.3) should have a plus sign.

By defining $r_\pm = m \pm \sqrt{m^2 + l^2}$ such that:

$$f(r) = \frac{(r - r_+)(r - r_-)}{r^2 + l^2} \tag{4.4}$$

we find some kind of singularity for $f(r) = 0$ at $r = r_\pm$. In fact, these two hypersurfaces correspond to Killing horizons generated by ∂_t.

For $r < r_+$ and $r > r_+$, $f(r) > 0 \quad \Rightarrow \quad r$ is spacelike and t is timelike - the metric is stationary. These regions are called NUT_- and NUT_+ respectively.

For $r_- < r < r_+$, $f(r) < 0 \quad \Rightarrow \quad t$ is spacelike and r is timelike. This time-dependent region is called the Taub region.

Some initial interpretations of the string singularity identified the solution with the natural idealization of a semi-infinite massless source of angular momentum[19, 20], where the source would be a thin semi-infinite spinning rod.

After noticing that in the linearized limit Taub-NUT gives the equivalent of a magnetic monopole, [21] draws the analogy between the NUT solution and Dirac's theory of magnetic monopoles. It regards the source at the origin as a dyon - an ordinary mass (quantified by parameter m) together with a magnetic mass (quantified by the NUT parameter l), such that the string singularity in the metric is analogous to the Dirac string of the magnetic monopole.

Additionally, [22] supports the analogy by showing that a test mass in the stationary (NUT) region possesses the same properties as those of an electrically charged particle orbiting a magnetic monopole. The case for the gravitational analogue of a magnetic monopole on an electrically charged particle is yet to be made.

A curious contradiction, is the fact that since (4.3) has no central curvature singularity for $l \neq 0$, it is not clear where the source of the field is located.

Returning to the string singularity, Misner[23] got around this problem by choosing two charts with differing time coordinates.

For $0 < \theta < \pi/2$, the metric is the previous (4.3), and for $\pi/2 < \theta < \pi$, we introduce $t \to \tilde{t} = t - 4l\phi$ such that the two metrics join smoothly at $\theta = \pi/2$. Since $\phi = 0$ and $\phi = 2\pi$ are identified in our coordinate system, that is $\phi \sim \phi + 2\pi$, for consistency our new time coordinate must have a periodicity of $\Delta\tilde{t} = 8\pi l$.

Thus, we manage to get rid of our string singularity, at the cost of a periodic time coordinate, where all timelike curves that move along coordinate time are closed (closed timelike curves)[3]. In fact, this periodicity is equivalent to the Dirac quantization condition[21].

However, in this interpretation, there seems to be no argument to consider the NUT parameter as a magnetic mass. At most, it is related to the periodicity of the closed timelike curves.

4.2 Quantization condition

As previously mentioned, Taub-NUT is a full solution of Einstein's equations that contains duality invariance[4]. Therefore, it is a good example of what the spacetime surrounding a gravitational monopole could be, with the closed timelike curves of Misner's interpretation presenting the analogous Dirac quantization condition.

This is extended by [14]. Although they are able to introduce external electric and magnetic energy-momentum tensors into the spin-2 (i.e. gravity) linearized theory (see section 3.2), the formulation of external sources that are covariantly conserved in the non-linear regime is still incomplete. So in order to learn more about the non-linear case, they develop their own way of obtaining the quantization condition, and proceed to calculate the Poincaré charges of the Taub-NUT solution.

[3] This is only a problem in the NUT regions where t and $\tilde{t}$ are timelike.

[4] In the linearized limit, the tools from section 3.2 can be used to show that the Riemann tensor for $l = 0$, $m \neq 0$ is dual to $l \neq 0$, $m = 0$.

First, the Killing vectors for our metric (4.3):

$$\xi_t = \frac{\partial}{\partial t} \tag{4.5}$$

$$\xi_x = -\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi} + \left(2l\cos\phi\cot\theta - 2l\frac{\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial t} \tag{4.6}$$

$$\xi_y = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi} + \left(2l\sin\phi\cot\theta - 2l\frac{\sin\phi}{\sin\theta}\right)\frac{\partial}{\partial t} \tag{4.7}$$

$$\xi_z = \frac{\partial}{\partial\phi} - 2l\frac{\partial}{\partial t} \tag{4.8}$$

For each of these, the components perpendicular to the time direction look exactly like those for a 3d space with spherical symmetry:

$$\xi_X = -\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi} \tag{4.9}$$

$$\xi_Y = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi} \tag{4.10}$$

$$\xi_Z = \frac{\partial}{\partial\phi} \tag{4.11}$$

But what about the time direction components? Well, if to this space we add a monopole vector potential, in order for the potential to stay invariant under active rotation transformations, we also need to do an appropriate gauge transformation such that $\delta_{\xi_B} A_i = \mathcal{L}_{\xi_B} A_i + \partial_i \Lambda_B = 0$.

Once these are found, applying everything is equivalent to adding a new coordinate to the Killing vector fields, (e.g. a new $\partial/\partial\lambda$ direction), and putting the gauge transformation there:

$$\hat{\xi}_X = -\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi} + \left(2l\cos\phi\cot\theta - 2l\frac{\cos\phi}{\sin\theta}\right)\frac{\partial}{\partial\lambda} \tag{4.12}$$

$$\hat{\xi}_Y = \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi} + \left(2l\sin\phi\cot\theta - 2l\frac{\sin\phi}{\sin\theta}\right)\frac{\partial}{\partial\lambda} \tag{4.13}$$

$$\hat{\xi}_Z = \frac{\partial}{\partial\phi} - 2l\frac{\partial}{\partial\lambda} \tag{4.14}$$

where λ is the conjugate to the generator of $U(1)$ transformations, and the coordinate on the $U(1)$ fibres of the manifold.

In this case, the rotationally invariant, monopole one-form field looks like:

$$A = 2l(1 - \cos\theta)d\phi \tag{4.15}$$

where this is really in the coordinate basis of the chart with Dirac string along the negative z-axis.

Thus, we can notice that for (4.3), the Killing vector fields have the same components as those of a monopole with vector potential $A^i \sim g^{0i}$ [5].

In fact, for an infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$, $g_{0i} \to g_{0i} + \partial_0 \xi_i + \partial_i \xi_0$. Such that for $\xi_\mu = (\Lambda, 0, 0, 0)$, $\delta g_{0i} = \partial_i \Lambda$, and substituting our definition for the vector field, $A_i \to A_i + \partial_i \Lambda$. Therefore, diffeomorphisms in the time direction are equivalent to a gauge transformation of the vector potential, which is why the gauge parameter λ is the time direction for (4.5-4.8).

Indeed, the Killing vectors for Taub-NUT form the algebra:

$$[\xi_a, \xi_b] = -\epsilon_{abc}\xi_c \tag{4.16}$$

$$[\xi_a, \xi_t] = 0 \tag{4.17}$$

where a,b,$c \in \{x, y, z\}$.

From this we can see that the isometries of Taub-NUT are generated by an $\mathfrak{su}(2) \times \mathfrak{u}(1)$ Lie algebra, such that our spacetime is spherically symmetric (which upon initial inspection of the metric isn't explicitly clear) and stationary[6].

With each Killing vector field there is an associated charge that is conserved along geodesics; $Q = m_0 u_\mu \xi^\mu$, where m_0 is the rest mass of the particle following said geodesic, and u_μ its 4-velocity.

Thus, $P_z = m_0 u_\mu \xi_z^\mu = p_\phi - 2l m_0 u_0$ is one such conserved quantity.

In classical electromagnetism, the additional component is due to the angular

[5] In the weak field limit, $A^i = g^{0i}$

[6] ξ_t defines a Killing vector which is asymptotically timelike, however not globally since it generates the two Killing horizons $r_\pm$

momentum of the electromagnetic field, which in this analogous situation is $2lm_0u_0$.

By requiring this angular momentum to be quantized in multiples of $\hbar/2$, which is known to give the Dirac quantization condition, we obtain the quantization condition:

$$4lm_0u_0 \in \hbar\mathbb{Z} \tag{4.18}$$

We can indeed confirm that this is the same quantization condition obtained by the periodicity of the time coordinate in Misner's interpretation - since the time dependence of the wavefunction, $\psi \propto e^{-\frac{iEt}{\hbar}}$, and the closed timelike curves are contractible, we require the wavefunction to be single-valued in their time period $\Delta t = 8\pi l$. Therefore, we obtain (again):

$$\frac{E\Delta t}{\hbar} = 2\pi\mathbb{Z} \tag{4.19}$$

$$4lm_0u_0 \in \hbar\mathbb{Z} \tag{4.20}$$

for $E = m_0u_0$, where the orbiting particle only has electric mass, m_0, and 4-velocity u_μ.

Resorting to the duality invariance of Taub-NUT, we conclude that this result would look the same for a magnetic mass in the gravitational field of an electric mass and vice versa.

Thus, the quantization condition (3.61) obtained in section 3.3 for the linearized theory, also holds for the non-linear theory:

$$\frac{4G(P_\gamma\bar{Q}^\gamma - \bar{P}_\gamma Q^\gamma)}{\hbar} \equiv \frac{4G\epsilon_{ab}Q^a_\gamma\bar{Q}^{b\gamma}}{\hbar} \in \mathbb{Z} \tag{4.21}$$

where the Minkowski metric is being used for the greek letter inner product.

It can be checked by comparing (4.21) with (4.18) for:

$$\bar{Q}^{1\gamma} = (m, 0, 0, 0) \tag{4.22}$$

$$\bar{Q}^{2\gamma} = (l, 0, 0, 0) \tag{4.23}$$

$$Q^1_\gamma = m_0 u_\gamma \tag{4.24}$$

$$Q^2_\gamma = 0 \tag{4.25}$$

where we are in the rest frame of the dyon $\bar{Q}^a_\gamma$.

In order to calculate the Poincaré charges via the treatment from [24], Taub-NUT has to satisfy their specific requirements of asymptotic flatness at spatial infinity.

As we stated previously, this solution is not globally asymptotically flat due to its string singularity - in spite of this, the aforementioned conditions are still satisfied.

Thus, as $r \to \infty$ the metric should approach Minkowski spacetime as:

$$\begin{aligned} h_{rr} = O(r^{-1}) \ ; \ h_{r\theta} = O(1) \ ; \ h_{r\phi} = O(1) \\ h_{\theta\theta} = O(r) \ ; \ h_{\theta\phi} = O(r) \ ; \ h_{\phi\phi} = O(r) \end{aligned} \tag{4.26}$$

where the leading terms (of the mentioned order) in h_{rr}, $h_{r\phi}$, $h_{\theta\theta}$, $h_{\phi\phi}$ should be even under the inversion $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$, while the leading terms of $h_{r\theta}$, $h_{\theta\phi}$ should be odd.

Likewise, for the conjugate momenta[7]:

$$\begin{aligned} \pi^{rr} = O(1) \ ; \ \pi^{r\theta} = O(r^{-1}) \ ; \ \pi^{r\phi} = O(r^{-1}) \\ \pi^{\theta\theta} = O(r^{-2}) \ ; \ \pi^{\theta\phi} = O(r^{-2}) \ ; \ \pi^{\phi\phi} = O(r^{-2}) \end{aligned} \tag{4.27}$$

with leading terms in π^{rr}, $\pi^{r\phi}$, $\pi^{\theta\theta}$, $\pi^{\phi\phi}$ odd, and those in $\pi^{r\theta}$, $\pi^{\theta\phi}$ even.

These conditions are clearly satisfied for the metric (4.1) in the coordinate system

[7] They are defined as the conjugate momenta to the spatial components of the metric, for the Taub-NUT Lagrangian. See ADM formalism.

with $k = 0$, but not any other value, as otherwise they break the parity conditions.

Also, because these conditions are satisfied, we can use the Minkowski scalar product for equation (4.21).

Finally, [14] uses the surface integrals defined by [24] to calculate the electric generators of Poincaré transformations at null infinity, which indeed form a symmetry group of the system at the boundary due to the asymptotic behaviour of Taub-NUT which we verified.

These generators, and therefore conserved electric charges of the Taub-NUT spacetime, are:

$$P^0 = m \tag{4.28}$$

$$P^i = 0 \tag{4.29}$$

$$J_{ij} = 0 \tag{4.30}$$

$$J_{0i} = 0 \tag{4.31}$$

The first two are due to the translation invariance, hence (4.28) defines the energy and (4.29) the momentum. Whilst the last two are due to spatial rotational invariance, such that (4.30) gives the angular momentum of the system.

We may have concerned earlier because the Killing vectors depend on the magnetic mass, however, when we calculate the surface integrals these terms disappear due to the parity constraints. Therefore, it is very nice to see the magnetic mass doesn't contribute to the electric Poincaré charges.

Interestingly, the zero angular momentum supports the interpretation of a dyon source at the origin, which for the classical solution was a point of contention! (see previous section). If we separate the magnetic and the electric mass, we gain angular momentum. Such a solution belongs to what is called the Kerr-Newman-Taub-NUT family of solutions.

Chapter 5

Higher theory of gravity

For the last chapter, we take inspiration from the 't Hooft-Polyakov monopole to try to find the conditions for a gravitational monopole, from a higher theory.

5.1 Cartan geometry

A natural way to extend the concepts from TP monopoles to gravity can be found in the mathematics of Cartan geometry.[1]

The main concept behind its workings can be understood with the analogy of the waywiser; an instrument used to measure physical distances on a road, which consists of a wheel connected to a sick. As the wheel is rolled through a path without slipping, the distance is measured by counting how many times it has rotated. Expanding this idea to obtain more information about the road, we roll instead a sphere along a closed path and compare how much it has rotated by the end.

Usually, when we want to describe a two-dimensional Riemannian manifold we use an SO(2) connection, such that when we parallel transport a vector along a closed path, it can tell us how much the vector has rotated in its two-dimensional tangent space. We can extend this thought to the ball - to find out how much it has rotated in its embedding 3d space, we instead use an SO(3) connection, and instead of attaching tangent spaces at each point, we attach two-spheres. That is,

[1] For a detailed introduction, see [25, 26]

we describe a manifold by rotating on top of it (without slipping) a homogeneous space.

In fact, if the homogeneous space itself has more in common with the manifold, it is better suited to describing it. Just like how rotating a sphere over a slight deformation of another sphere is more useful than rotating it over a plane.

To describe Einstein-Cartan gravity in a Lorentzian manifold, we choose a de Sitter homogeneous spacetime with an $SO(1,4)$ connection A^{AB}. Inside this connection are the degrees of freedom of the spin-connection and tetrad field. A way we can separate them out is by introducing a scalar field ϕ^A and defining a projector:

$$P^A{}_B = \delta^A_B - \frac{1}{\phi^C \phi_C} \phi^A \phi_B \tag{5.1}$$

where we use the indices $A, B, C \in \{0, 1, 2, 3\}$. From here, we can define the spin-connection and tetrad field:

$$e^A = P^A{}_C D\phi^C \tag{5.2}$$

$$\omega^{AB} = A^{AB} + \frac{2}{\phi^C \phi_C} \phi^{[A} e^{B]} \tag{5.3}$$

where $D\phi^A = d\phi^A + A^A{}_B \phi^B$.

If we break the symmetry of the scalar field in the gauge $\phi^A \overset{*}{=} \phi_0 \delta^A_4$ [2], which we can always find as long as $\phi^A \phi_A > 0$, with constant ϕ_0 we find the usual result from Cartan gravity:

$$A^{AB} = \begin{pmatrix} \omega^{ab} & \phi_0 e^a \\ -\phi_0 (e^a)^T & 0 \end{pmatrix} \tag{5.4}$$

[2] $\overset{*}{=}$ means the equality holds within a particular gauge.

with the usual SO(1,3) indices $a, b, c \in \{0, 1, 2, 3\}$. Such that:

$$e^A \overset{*}{=} \begin{pmatrix} e^a \\ 0 \end{pmatrix} \tag{5.5}$$

$$\omega^{AB} \overset{*}{=} \begin{pmatrix} \omega^{ab} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.6}$$

We can also define a field strength $F^{AB} = dA^{AB} + A^A{}_C \wedge A^{CB}$, such that if we rearrange equation (5.3) we obtain:

$$R^{AB} = F^{AB} + \frac{1}{\phi^C \phi_C} e^A \wedge e^B - \frac{2}{\phi^C \phi_C}\left(T^{[A}\phi^{B]} - \frac{1}{2} d\log(\phi^D \phi_D) e^{[A}\phi^{B]}\right) \tag{5.7}$$

for $R^{AB} = d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB}$. And in the previous gauge with broken symmetry:

$$R^{AB} \overset{*}{=} \begin{pmatrix} R^{ab} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.8}$$

To recover the dynamics of Einstein-Cartan gravity, we use the action:

$$S[A, \phi] = \int e_{ABCDE} \phi^E \wedge D\phi^A \wedge D\phi^B \wedge F^{CD} \tag{5.9}$$

with $\phi^A \phi_A = const > 0$. Since $e^a{}_\mu \overset{*}{=} \delta^a_B D_\mu \phi^B = D_\mu \phi^a$, with (5.8) the action above reduces to the Einstein-Cartan action up to a constant.

Now we are dealing with a situation much more similar to that of a gauge theory, since the degrees of freedom come only from a connection and scalar field with $SO(1, 4)$ local gauge invariance.

5.2 Finding 't Hooft

Taking a step back, we can use this formalism to describe the TP monopole from section 2.3.

Starting with an $SO(3)$ connection A^{AB} and scalar field ϕ^A, we can redefine the degrees of freedom using the SO(3) invariant Levi-Civita tensor:

$$A^A = \frac{1}{2}\epsilon^{ABC} A_{BC} \tag{5.10}$$

Then, we project the connection a bit differently to obtain:

$$\tilde{\omega}^{AB} = P^A_{\ C} P^B_{\ D} A^{CD} \tag{5.11}$$

where we can also define $\tilde{\omega}^{AB} = \frac{1}{2}\epsilon^{ABC}\Omega_C$.

Expanding everything out, we obtain:

$$\tilde{\omega}^{AB} = A^{AB} - \frac{2}{\phi^D \phi_D}\phi^{[B} A^{A]D}\phi_D \tag{5.12}$$

$$= \epsilon^{ABC} A_C - \frac{2}{\phi^D \phi_D}\phi^{[B}\epsilon^{A]EC} A_C \phi_E \tag{5.13}$$

such that:

$$\tilde{\omega}^A = A^A - \frac{1}{\phi^C \phi_C}\phi^B \phi_{[B} A_{C]} \tag{5.14}$$

$$= (\hat{\phi}^B A_B)\hat{\phi}^A \tag{5.15}$$

$$\tilde{\omega}^A_\mu = \mathcal{A}_\mu \hat{\phi}^A \tag{5.16}$$

where $\hat{\phi}^A = \frac{\phi^A}{\sqrt{\phi^B \phi_B}}$

This is the same as 't Hooft's definition (2.32) for the new gauge invariant vector field, $\mathcal{A}_\mu = \frac{\phi_A A^A}{\sqrt{\phi^B \phi_B}}$. Since $\hat{\phi}^A$ has unit magnitude, setting an $SO(4)$ gauge corresponds to choosing the direction in field space in which it breaks the symmetry. The remaining degree of freedom is due to the remaining $U(1)$ symmetry, and we can isolate it to find it is indeed $\tilde{\omega}^A_\mu \hat{\phi}_A = \mathcal{A}_\mu$.

For the field strength, we project like:

$$\mathcal{F}^{AB} = P^A_C P^B_D \big(F^{CD} + \frac{1}{\phi^D\phi_D} D\phi^C \wedge D\phi^D\big) \quad (5.17)$$

$$\mathcal{F}^A_{\mu\nu} = \frac{1}{\phi^C\phi_C} F^B_{\mu\nu}\phi_B\phi^A + \frac{1}{\phi^F\phi_F} P^B_C P^E_D \epsilon^A{}_{BE} D_{[\mu}\phi^C D_{\nu]}\phi^D \quad (5.18)$$

$$= (F^B_{\mu\nu}\hat{\phi}_B)\hat{\phi}^A + \frac{1}{\phi^F\phi_F}(\delta^B_C\delta^E_D - \frac{2}{\phi^G\phi_G}\delta^E_D\phi^B\phi_C)\epsilon^A{}_{BE} D_{[\mu}\phi^C D_{\nu]}\phi^D \quad (5.19)$$

and contracting with the field again to obtain the gauge invariant degree of freedom we obtain:

$$\mathcal{F}_{\mu\nu} = \mathcal{F}^A_{\mu\nu}\hat{\phi}_A \quad (5.20)$$

$$= F^B_{\mu\nu}\hat{\phi}_B + \frac{1}{\phi^F\phi_F}(\hat{\phi}_A\delta^B_C\delta^E_D - \frac{2}{\phi^G\phi_G}\delta^E_D\hat{\phi}^{[A}\phi^{B]}\phi_C)\epsilon_{ABE} D_{[\mu}\phi^C D_{\nu]}\phi^D \quad (5.21)$$

$$= F^A_{\mu\nu}\hat{\phi}_A + \frac{1}{(\phi^D\phi_D)^{3/2}}\epsilon_{ABC}\phi^A D_\mu\phi^B D_\nu\phi^C \quad (5.22)$$

which is exactly equation (2.31)!

Thus, we have obtained again the gauge invariant re-definitions for the field strength and connection.

5.3 Breaking $SO(1,4)$

Turning to gravity, we proceed by describing it as a broken $SO(1,4)$ gauge theory, as we learn at the beginning of this chapter. We have a scalar field ϕ^A and connection A^{AB}, but this time we project the connection as for the TP monopole:

$$\tilde{\omega}^{AB} = P^A{}_C P^B{}_D A^{CD} \quad (5.23)$$

But how does this compare to the projection that gives the spin-connection degrees of freedom from equation (5.3)? Well, we can calculate their difference to find:

$$\omega^{AB} - \tilde{\omega}^{AB} = A^{AB} + \frac{2}{\phi^C\phi_C}\phi^{[A}e^{B]} - A^{AB} + \frac{2}{\phi^D\phi_D}\phi^{[B}A^{A]D}\phi_D \tag{5.24}$$

$$= \frac{2}{\phi^C\phi_C}\phi^{[A}d\phi^{B]} \tag{5.25}$$

which means:

$$\tilde{\omega}^{AB} \stackrel{*}{=} \omega^{AB} \tag{5.26}$$

Thus, if the scalar field breaks in the manner $\phi^A = \phi_0\delta_4^A = const > 0$ we have the Bianchi identity:

$$D^{(\tilde{\omega})}R^{AB} = 0 \tag{5.27}$$

But what if they are not equal? In that case, if we consider $\omega^{AB} - \tilde{\omega}^{AB} = Y^{AB}$, we have by definition:

$$D^{(\tilde{\omega})}R^{AB} = dR^{AB} + \tilde{\omega}^A{}_C \wedge R^{CB} - R^A{}_C \wedge \tilde{\omega}^{CB} \tag{5.28}$$

$$= D^{(\omega)}R^{AB} - Y^A{}_C \wedge R^{CB} + R^A{}_C \wedge Y^{CB} \tag{5.29}$$

where, by definition, $D^{(\omega)}R = 0$. Finally, setting $Y^{AB} = \frac{2}{\phi^C\phi_C}\phi^{[A}d\phi^{B]}$ we find:

$$D^{(\tilde{\omega})}R^{AB} = -\frac{2}{\phi^D\phi_D}\phi^{[A}d\phi^{C]} \wedge R_C{}^B - \frac{2}{\phi^D\phi_D}\phi^{[B}d\phi^{C]} \wedge R^A{}_C \tag{5.30}$$

which for $\phi^{[A}d\phi^{C]} \neq 0$ breaks the Bianchi identity, and thus corresponds to some kind of gravitational magnetic current!

Chapter 6

Conclusion

In this dissertation, we gave an overview of the different areas which tackle the concept of monopoles in gravity. After introducing the electromagnetic monopole, as defined by Dirac and 't Hooft, we showed how Zee used the similarity between Maxwell's equations and linearized gravity to define a gravipole. Then, for a more general treatment, we reviewed [14], which gives the general quantization condition for a duality invariant spin-2 gauge theory, in other words linearized gravity. Of course, since this all of this only works in the weak-field approximation, we are left with the question of how a gravity monopole could be defined in the highly non-linear theory of general relativity.

This gives way to a solution of the Einstein field equations called Taub-NUT. It is duality invariant, and in one of its interpretations, it describes the spacetime around a dyon - a point particle with the usual/electric mass as well as its dual/magnetic counterpart. We point out the unnoticed fact that, by considering the Poincaré charges of the manifold, [14] gives very good support for the interpretation that Taub-NUT describes a spacetime with a source at the origin, rather than a rod of infinite angular momentum - a point of contention in some of the literature.

Although Taub-NUT correctly reduces to Zee's gravipole in the weak-field approximation, we invoke the spirit of Zee's original paper, and consider another alternative: that the gravipole might actually be a part of a higher-theory, in a manner akin to the 't Hooft-Polyakov monopole. Doing so, we use Cartan geometry to reproduce 't Hooft's famous monopole solution, and then extend it to consider gravity as a spontaneously broken $SO(1,4)$ gauge theory coupled to a scalar field in the fundamental representation, which in the broken state breaks the Bianchi identity. More precisely, the conditions which have to be satisfied for the scalar field to break Bianchi are laid out.

It is important to note that $\pi_3(SO(5)/SO(4))$ is trivial[1], as well as the lower homotopy groups. This suggests that regardless of condition (5.30), soliton solutions cannot be found for this theory. However, the techniques described in chapter 5 are still interesting, and further work could involve considering a broken $SO(1,4)$ theory with five-dimensional spacetime, since this would allow for a gravipole, because for the boundary of such a spacetime, $\pi_4(SO(5)/SO(4))$ is not trivial and allows for solitons.

It would also be interesting to expand the relationship between torsion and monopoles to the non-linear case, which was pointed out in section 3.2. Especially with the mathematics of Cartan gravity, since it seems that in the linearized regime, adding externally a dual energy-momentum tensor goes hand-in-hand with including torsion degrees of freedom.

[1]Complexify time to go from $SO(1,4)$ to $SO(5)$

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